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Analogy Between the Lorenz Strange Attractor and a Bistable Stochastic Oscillator

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The relation between the aperiodic solution of the Lorenz model and that of a stochastic anharmonic oscillator is explored. The stochastic oscillator is constructed by replacing $\dot{z}(t)$ in the Lorenz model by a stochastic variable $\zeta(t)$ of specified statistics. The resulting system is of course not isomorphic to the Lorenz model, but does share with it a number of statistical properties. Thus, within the confines of these measures the two systems are physically very similar.

KEY WORDS: Lorenz model; strange attractor; bistable stochastic oscillator.

1. INTRODUCTION

Hydrodynamic systems can exist in one of three pure states of motion: steady-state flows, regular periodic flows, and irregular or aperiodic flows. The first two types of motion occur most often in controlled laboratory experiments. The last type of motion has resisted predictive methods, and long-time observations indicate that the flow patterns do not repeat themselves. Aperiodic flows are quite common in geophysical contexts, with the resultant uncertainty in the weather. They have also recently been manifest in controlled laboratory experiments of *deterministic* flow fields. For example, Ahlers and Walden[(1)] demonstrated experimentally that a Rayleigh–Bénard system exhibits a transition between spatially ordered periodic states. During this transition the fluid appears turbulent. The time interval between states of turbulent activity is random, as is the duration of the turbulent burst. Ahlers and Walden interpret their experimental results in

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terms of the random transitions between the minima of a bistable potential. An alternative to this kind of description may be the deterministic but aperiodic motion of a strange attractor solution to truncated forms of the hydrodynamic equations, as first constructed by Lorenz.[2]

Lorenz developed his model by severely truncating the hydrodynamic equations (from an infinite number of modes to three modes) describing Bénard convection in the atmosphere. His interest was in the feasibility of very long-range weather prediction when such nonperiodic flows are possible. Other investigators have found that the aperiodic behavior observed by Lorenz is also present in other dissipative nonlinear systems. For example, a set of equations mathematically equivalent to the Lorenz system arises in the analysis of laser problems leading to the possibility of optical chaos.[3,4] For certain values of the parameters the Lorenz system yields three steady states. The trajectory consists of nonperiodic circulation around two of these states and transitions between them (Fig. 1). These

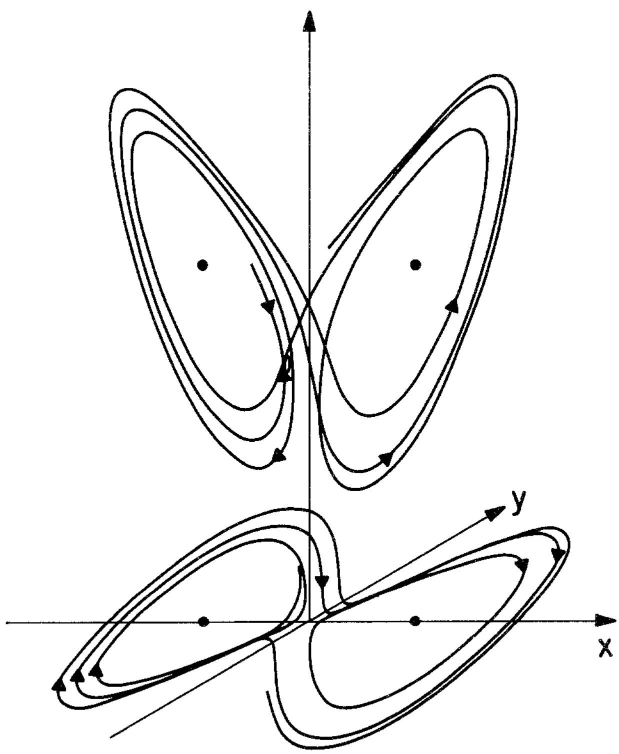

Fig. 1. Trajectory of the Lorenz model ($r = 30$, elapsed time 4.5) projected onto (bottom) the x–y plane and (top) onto the x–z plane. The points are the steady-state solutions (2.3). Taken from Ref. 6.

cyclic motions and transitions occur without any simple regularity or order.[2]

The irregular behavior of the trajectories has motivated the study of the Lorenz model using statistical methods.[5–9] Numerical studies have shown that the time averages of the state variables are independent of the initial conditions, even though the actual trajectory is very sensitive to these conditions.[3] Guided by this knowledge, Lücke[6] undertook an analysis of the statistical properties of the three state variables and calculated their moments and correlation functions. Zippelius and Lücke[7] added white noise terms to the dynamical equations and studied the Lorenz model as a stochastic problem. Aizawa[8] viewed the Lorenz dynamics *itself* as a stochastic process (on an appropriate time scale) without adding any external noise. He selected the x variable as an example for his study, and separated the process into two components. One component represents the quasiperiodic variations of the motions around either of the steady states. The other is the "flip-flop" process, i.e., the random switching between the two sets of quasiperiodic motions. By fitting the dynamics to a superposition of these two kinds of stochastic processes, he was able to calculate the probability distribution for x and for the persistence time in the vicinity of one of the steady states.

The irregular or aperiodic solutions of the type found in the Lorenz model have by now been found in other few-degree-of-freedom systems and have been recognized as one possible *signature of nonlinear behavior.* Qualitatively analogous behavior has long been recognized as the *signature of fluctuations* driving physical systems that are in contact with their surroundings (i.e., systems that are not isolated). The latter view has been basic to the field of statistical mechanics, where it is always assumed that fluctuations and the resultant stochastic behavior of the system arise from a large number of unresolved degrees of freedom with which the system of interest interacts. The language often used to describe the dynamics of statistical mechanical systems is that of stochastic differential equations, wherein the fluctuations that arise from the large number of unresolved degrees of freedom appear as noise terms of specified statistics.

The qualitative analogy between the irregular solutions of few-degree-of-freedom systems and the stochastic solutions of many-degree-of-freedom systems leads us to explore a possible relation between them in this paper. In particular, we wish to explore the possibility of associating the noise in a stochastic differential equations with the irregular behavior of a single variable in a few-degree-of-freedom system. If such a relation can be established, then one may tentatively conclude that the sources of noise in statistical mechanical systems may not always correspond to *many* unresolved degrees of freedom.

Knobloch[9] constructed a second-order stochastic differential equation (SDE) by considering the rapidly changing state variable z to be a fluctuating quantity with specified statistical properties. His equation is analogous to that of the displacement of a stochastically driven and linearly damped "oscillator." The system described by this equation is unstable except for a single particular ratio of parameter values (cf. Section 2).

In this paper we pursue the idea originated by Knobloch[9] and present a different stochastic analog for the Lorenz model which possesses a number of desirable qualitative properties. The SDE derived here is of the form of a *nonlinear* oscillator of the Van der Pol–Duffing type[10] driven by multiplicative noise.[11,12] The potential involved in this SDE has three extrema corresponding to the three steady states of the Lorenz model.

In Section 2 we present the actual reduction of the Lorenz equations to a SDE for an anharmonic oscillator and make the correspondence between the Lorenz and oscillator dynamics. We also compare our equations with those derived by Knobloch.[9] In Section 3 we compute the energy envelope[11,12] of the oscillator and compare certain second-order quantities with those obtained via numerical simulations of the Lorenz system.[6] We present estimates of the mean persistence time, i.e., the average time that a trajectory spends near a steady state, using the oscillator SDE. The last section contains some concluding remarks.

2. THE OSCILLATOR ANALOGY

Lorenz analyzed the strange attractor nature of his model in the context of a numerical study of the Bénard convection problem.[2] The nonperiodic nature of the solution had been previously observed by Saltzman.[13] The Lorenz equations can be written in the dimensionless form

$$\dot{x} = \sigma y \tag{2.1a}$$

$$\dot{y} = (r-1)x - (\sigma+1)y - xz \tag{2.1b}$$

$$\dot{z} = -bz + xy + x^2 \tag{2.1c}$$

Lücke[6] obtained this version of the model by the change of variable $y^* - x = y$ in the original version. Here σ is the Prandtl number and r is the ratio of the Rayleigh number to the critical Rayleigh number. The variable x is the Fourier component of the gravest velocity mode; y^* represents the horizontal dependence of the temperature field; and z represents the deviation of the vertical temperature variation from the linear profile corresponding to steady conduction.

The system (2.1) always has the steady-state solution

$$x = y = z = 0 \tag{2.2}$$

corresponding to steady conduction. For $r < 1$ this is the only steady state and it is stable. At $r = 1$ a bifurcation occurs. For $r > 1$, the state given by (2.2) is unstable and two more steady states emerge, which are stable. These new states are given by

$$x = \pm[b(r-1)]^{1/2}, \qquad y = 0, \quad z = r - 1 \tag{2.3}$$

They correspond to steady rolls and remain stable for $1 < r < r_T$, where

$$r_T = \sigma \frac{\sigma + b + 3}{\sigma - b - 1} \tag{2.4}$$

At $r = r_T$ a second bifurcation occurs. This is the onset of the irregular motion that has been associated with turbulence.[14] For $r > r_T$ linear stability analysis indicates that all the above steady states are locally unstable. Since there is no stable steady state in this parameter regime, time-dependent solutions such as limit cycles are expected. However, the presence of limit cycles can be demonstrated only for certain values of $r > r_T$. For other values of r the trajectory is nonperiodic and wanders around in the vicinity of the three unstable steady states (Fig. 1).[2,6] The trajectory is deflected when approaching the state given by Eq. (2.2), whereas it is somewhat attracted by and circles around the states given by Eq. (2.3). Mathematically this behavior is a manifestation of the fact that the fixed point (0, 0, 0) has a one-dimensional unstable manifold, whereas the two other fixed points have two-dimensional unstable manifolds.

In the above regime the Lorenz trajectory visits almost every point in certain domains of phase space. Based on this observation, Knobloch[9] interpreted the Lorenz dynamics as an ergodic (stochastic) process in which all three state variables are random functions of time. Further, he eliminated one of the state variables, $z(t)$, from the equations and replaced it with an external noise. This procedure assumes that the evolution of z is independent of x and y. Specifically, he defined $\omega(t)$ as

$$\omega(t) = \sigma(z - z_0) \tag{2.5}$$

where z_0 is the time-average of z obtained from numerical simulations. Interpreting $\omega(t)$ as a fluctuating quantity with specified statistics, he derived the equations

$$\dot{x} = \sigma y \tag{2.6a}$$

$$\dot{y} = Ax - By - x\omega(t) \tag{2.6b}$$

where

$$A = \sigma(r - 1 - z_0) > 0 \tag{2.7}$$

and

$$B = \sigma + 1 \tag{2.8}$$

The inequality in (2.7) is derived by Knobloch.[9] Equations (2.6) describe a mechanical system moving under the influence of the potential

$$\hat{V}(x) = -\tfrac{1}{2}Ax^2 \tag{2.9}$$

and also driven by a fluctuating force and linear dissipation. The potential $\hat{V}(x)$ is always negative and has a maximum at $x = 0$. It is well known that such a system is globally unstable in the absence of fluctuations. In the presence of fluctuations possessing a particular nonzero correlation time this global instability is suppressed. This is the value chosen by Knobloch in his analysis.

Looking at the projections of the trajectories on the x–y plane (Fig. 1) and the steady states of the Lorentz model, we find them to be analogous to the trajectories and equilibrium states of a mechanical oscillator in a *bistable* potential.[15] The trajectories of the undamped deterministic oscillator are closed orbits corresponding to both global and local oscillations. In other words, some orbits enclose only one of the two minima of the potential and others enclose all three extrema. If in addition the oscillator is driven by fluctuations and dissipation, then one would expect the oscillator to shift from one localized orbit to another at irregular times and thus mimic the behavior of the Lorenz system. We can arrive at such an equation for $\dot{x}$ and $\dot{y}$ if we take $\dot{z}$ (instead of z) as a fluctuating quantity. Specifically, we replace $\dot{z}$ by $b\zeta(t)$, where $\zeta(t)$ is a stationary random process:

$$\dot{z} \to b\zeta(t) \tag{2.10}$$

We stress that (2.10) is not meant to imply that $\dot{z}(t)$ and $\zeta(t)$ necessarily share the same statistical behavior. Rather, we wish to explore whether the replacement (2.10) leads to a model that shares certain moment properties with the Lorenz model. Then Eq. (2.1c) yields

$$z = \frac{1}{b}xy + \frac{1}{b}x^2 - \zeta(t) \tag{2.11}$$

Substituting this into (2.1b), we obtain the set of equations

$$\dot{x} = \sigma y \tag{2.12a}$$

$$\dot{y} = (r-1)x - \frac{1}{b}x^3 - (\sigma + 1)y - \frac{1}{b}x^2y + x\zeta(t) \tag{2.12b}$$

Equations (2.12) describe a damped anharmonic mechanical oscillator with the potential[15]

$$V(x) = -\frac{1}{2}(r-1)x^2 + \frac{1}{4b}x^4 \tag{2.13}$$

and driven by multiplicative noise. For $r<1$ this potential has a single minimum at $x=0$ and therefore the stationary distribution of Eq. (2.12) is expected to be unimodal and peaked at $x=y=0$, the state corresponding to steady conduction in the Bénard problem. When $r=1$ the anharmonic oscillator undergoes a bifurcation. For $r>1$, $V(x)$ has a maximum at $x=0$ and two minima. The locations of these minima correspond precisely to the states given by Eq. (2.3). However, the oscillator states remain stable for all values of $r>1$, unlike those of the Lorentz system. Thus, the second bifurcation occurring in the Lorenz model at r_T does not occur in the oscillator equations (2.12). Nevertheless, we will see that the two systems share some interesting qualitative features.

The Lorenz equations (2.1) have two symmetry properties that constrain the trajectories. Lücke[6] made use of these symmetry properties to derive relations between the time-averaged moments of the variables x, y, and z. The set (2.1) possesses the property of time translational invariance. This symmetry is also a property of the ensemble represented by Eq. (2.12) and is manifest in the stationarity of the SDE. In other words, Eqs. (2.1) are autonomous differential equations and Eqs. (2.12) are autonomous stochastic differential equations because $\zeta(t)$ has been chosen to be stationary. Another important property is the invariance under the parity operation

$$(x, y, z) \rightarrow (-x, -y, z) \tag{2.14}$$

satisfied by the Lorenz equations. The oscillator equations also possess this symmetry.

Starting from the Lorenz equations (2.1) and using these symmetry properties, Lücke[6] obtained the following relationships between the time-averaged moments. If we denote $\langle x^k y^l z^m \rangle$ by $\langle k, l, m\rangle$, then

$$\begin{aligned} &k\sigma\langle k-1, l+1, m\rangle \\ &\quad + l[(r-1)\langle k+1, l-1, m\rangle - (\sigma+1)\langle k, l, m\rangle - \langle k+1, l-1, m+1\rangle] \\ &\quad + m[\langle k+1, l+1, m-1\rangle + \langle k+2, l, m-1\rangle - b\langle k, l, m\rangle] = 0 \end{aligned} \tag{2.15}$$

These relations do not form a closed set and therefore they cannot be solved completely. However, Lücke was able to obtain several simple relations between the low-order moments:

$$\langle x^k y \rangle = 0, \qquad k = 0, \pm 1, \ldots \tag{2.16a}$$

$$\langle x^2 \rangle = b \langle z \rangle \tag{2.16b}$$

$$\langle xyz \rangle = -(\sigma + 1)\langle y^2 \rangle \tag{2.16c}$$

$$\langle x(x+y)z \rangle = b\langle z^2 \rangle \tag{2.16d}$$

$$\langle x^2 z \rangle = \sigma \langle y^2 \rangle + (r-1)\langle x^2 \rangle \tag{2.16e}$$

$$b\langle x^2 z \rangle = 2\sigma \langle xyz \rangle + \langle x^4 \rangle \tag{2.16f}$$

We can derive similar relations between the moments of x, y, *and* z if we augment the oscillator equations (2.12) with Eq. (2.1c) for $\dot{z}$ and consider the resulting set as a system of SDEs in three variables. The stationary moments of this system satisfy

$$\begin{aligned} k\sigma\langle k-1, l+1, m \rangle &+ l\Bigg[(r-1)\langle k+1, l-1, m \rangle - \frac{1}{b}\langle k+3, l-1, m \rangle \\ &- (\sigma+1)\langle k, l, m \rangle - \frac{1}{b}\langle k+2, l, m \rangle \\ &+ \langle x^{k+1} y^{l-1} z^m \zeta(t) \rangle \Bigg] + m[-b\langle k, l, m \rangle \\ &+ \langle k+1, l+1, m-1 \rangle + \langle k+2, l, m-1 \rangle] = 0 \end{aligned} \tag{2.17}$$

Certain special cases (e.g., $l=0$, $m=0$) of Eq. (2.17) correspond to the exact equations (2.16a), (2.16b), (2.16d), (2.16f). We are not able to obtain (2.16c) and (2.16e) from Eq. (2.17).

Our purpose is to compare the time-averaged moments of the state variables of the Lorenz model and the corresponding *ensemble* average of the *stationary* process described by Eq. (2.12). If exact agreement is desired, the statistical properties of $\zeta(t)$ must of course exactly match those of $(1/b)\,\dot{z}(t)$ [cf. Eq. (2.10)]. Equation (2.12) is a set of *nonlinear* SDEs. Its corresponding stationary distribution cannot be obtained in closed form even under simplifying assumptions about the statistics of $\zeta(t)$. However, if $\zeta(t)$ is taken to be a delta-correlated Gaussian process (white noise), then one can use various approximation techniques to find analytic measures for the comparison (cf. Section 3).

If $\zeta(t)$ is taken to be white noise, then Eq. (2.12) is a special case of the Van der Pol–Duffing oscillator studied by Wiesenfeld and Knobloch.[11] They studied a general problem of the form (2.12) with arbitrary coefficients and they determined the conditions under which a stationary distribution can exist. Equation (2.12) satisfies the conditions they established for the existence of a normalizable stationary distribution. Wiesenfeld and Knobloch constructed the stationary distribution for weak damping. When the potential is bistable, the distribution is bimodal, as expected, with a saddle point at the origin. This behavior correctly depicts the structure of the projected trajectories of the Lorenz model.

3. ENERGY ENVELOPE AND PERSISTENCE TIME

In this section we take $\zeta(t)$ to be a Gaussian random function satisfying

$$\langle \zeta(t) \rangle = 0 \tag{3.1}$$

and

$$\langle \zeta(t)\, \zeta(t-\tau) \rangle = 2D\delta(\tau) \tag{3.2}$$

We calculate the statistical properties of the energy envelope[11,12] for the oscillator and compare the results with the corresponding quantities in the Lorenz model. The probability distribution $P(x, y, t)$ for the displacement x and momentum y of the oscillator satisfies the Fokker-Planck equation[16]

$$\frac{\partial}{\partial t} P(x, y, t) = \left\{ -\sigma y \frac{\partial}{\partial x} - \left[(r-1)x - \frac{1}{b} x^3 \right] \frac{\partial}{\partial y} \right.$$
$$\left. + \left[(\sigma + 1) + \frac{1}{b} x^2 \right] \left(1 + y \frac{\partial}{\partial y} \right) + Dx^2 \frac{\partial^2}{\partial y^2} \right\} P(x, y, t) \tag{3.3}$$

The stationary distribution is obtained by setting the right-hand side of Eq. (3.3) to zero. The resulting partial differential equation cannot be solved analytically, and even obtaining a numerical solution is difficult. Instead, we derive an approximate equation for the energy envelope of the oscillator and compute its average. Equation (2.12) in (x, y) space is transformed to (x, E) space by the change of variable[17]

$$E(x, y) = \tfrac{1}{2}\sigma y^2 + V(x) \tag{3.4}$$

The SDE then becomes

$$\text{(S)} \quad \dot{x} = \{2\sigma[E - V(x)]\}^{1/2}$$
$$\dot{E} = -2\left(\sigma + 1 + \frac{1}{b} x^2\right)[E - V(x)] + x\{2\sigma[E - V(x)]\}^{1/2} \zeta(t) \tag{3.5}$$

where the (S) in Eq. (3.5) indicates that the equation has been written using the Stratonovich stochastic calculus.[17]

The Fokker–Planck equation corresponding to Eq. (3.5) is

$$\frac{\partial}{\partial t} W_1(x, E, t) = \left\{-(2\sigma)^{1/2} \frac{\partial}{\partial x} [E - V(x)]^{1/2} + 2\left(\sigma + 1 + \frac{x^2}{b}\right) \frac{\partial}{\partial E} [E - V(x)] - D\sigma x^2 \frac{\partial}{\partial E} + 2\sigma D x^2 \frac{\partial^2}{\partial E^2} [E - V(x)]\right\} W_1(x, E, t) \tag{3.6}$$

where $W_1(x, E, t)\, dx\, dE = P(x, y, t)\, dx\, dy$. The joint probability density $W_1(x, E, t)$ can be written as the product

$$W_1(x, E, t) = W_2(x, t \mid E)\, W(E, t) \tag{3.7}$$

in terms of the conditional probability density $W_2(x, t \mid E)$ that the oscillator displacement is x at time t given that its energy envelope is E and the probability $W(E, t)$ that the energy envelope is E at time t irrespective of the displacement. Equation (3.6) is an equivalent representation of Eq. (2.12). To proceed further, we invoke an argument originally due to Stratonovich that $W_2(x, t \mid E)$ is proportional to the time spent at x by the oscillator whose energy envelope is E. The time spent at x is in turn inversely proportional to the velocity at x:

$$W_2(x, t \mid E) \propto \frac{1}{[E - V(x)]^{1/2}} \tag{3.8}$$

This approximation is valid if the circulation of the phase point within a given orbit is much faster than the shifting between orbits induced by fluctuations and dissipative effects. We assume that this description applies in the context of the Lorenz dynamics because orbits are clearly discernible in Fig. 1. The joint probability density $W_1(x, E, t)$ can then be written as

$$W_1(x, E, t) = \frac{W(E, t)}{2\phi'(E)[E - V(x)]^{1/2}} \tag{3.9}$$

where

$$\phi'(E) = \frac{1}{2} \int_R [E - V(x)]^{-1/2}\, dx \tag{3.10}$$

and where the region of integration R includes all values of x for which $E \geqslant V(x)$. There are two disjoint integration regions for $E < 0$. Substituting

(3.9) into the Fokker–Planck equation (3.6) and integrating over x yields the single variable equation

$$\frac{\partial}{\partial t} W(E,t) = \frac{\partial}{\partial E} \frac{[(\sigma+1)\,\phi(E) + \psi(E)/b - \sigma D \psi'(E)]}{\phi'(E)} W(E,t) + \sigma D \frac{\partial^2}{\partial E^2} \frac{\psi(E)}{\phi'(E)} W(E,t) \tag{3.11}$$

where $\phi(E)$, $\psi(E)$, and $\psi'(E)$ are defined by

$$\phi(E) \equiv \int_\alpha^\beta [E - V(x)]^{1/2}\, dx \tag{3.12}$$

$$\psi(E) \equiv \int_\alpha^\beta x^2 [E - V(x)]^{1/2}\, dx \tag{3.13}$$

$$\psi'(E) \equiv \frac{1}{2} \int_\alpha^\beta \frac{x^2}{[E - V(x)]^{1/2}}\, dx \tag{3.14}$$

where

$$\alpha = \begin{cases} \{b(r-1) - [b^2(r-1)^2 + 4bE]^{1/2}\}^{1/2} & \text{if } E < 0 \\ 0 & \text{if } E \geqslant 0 \end{cases} \tag{3.15}$$

$$\beta = \{b(r-1) + [b^2(r-1)^2 + 4bE]^{1/2}\}^{1/2} \tag{3.16}$$

Equation (3.11) can be solved for the stationary distribution $W_s(E)$ of the energy envelope:

$$W_s(E) = \frac{1}{N\sigma D}\, \phi'(E)\, e^{-q(E)/D} \tag{3.17}$$

where

$$q(E) \equiv \frac{1}{\sigma} \left[\frac{E}{b} + (\sigma+1) \int^E \frac{\phi(E')}{\psi(E')}\, dE' \right] \tag{3.18}$$

and N is a normalization constant.

We have calculated the distribution given by Eq. (3.17) for various values of D and the corresponding average energy envelope $\langle E \rangle$ and below we compare these with the results reported by Lücke.[6] He has plotted the first and second moments of z. All other moments appearing in Eq. (2.16) can be expressed in terms of these two. In particular,

$$\langle E\rangle_L = \frac{\sigma}{2}\langle y^2\rangle - \frac{1}{2}(r-1)\langle x^2\rangle + \frac{1}{4b}\langle x^4\rangle$$
$$= \left[\frac{3\sigma}{4} + \frac{\sigma(\sigma+1)}{2b} - \frac{1}{4}\right] b(r-1)\langle z\rangle - \left(\frac{3}{4} + \frac{\sigma+1}{2b}\right)\sigma b\langle z^2\rangle \qquad (3.19)$$

The values of $\langle z\rangle$ and $\langle z^2\rangle$ can be obtained from the figures in Ref. 6 and used to calculate $\langle E\rangle_L$. The resulting value can be compared with $\langle E\rangle$ obtained from an ensemble average. Alternatively, since the accuracy of the values thus read is limited, we replace $\langle E\rangle_L$ by $\langle E\rangle$ in (3.19) and attempt to reproduce $\langle z\rangle$ and $\langle z^2\rangle$. Equation (3.19) yields only one relation between $\langle z\rangle$ and $\langle z^2\rangle$, so that we need a second relation between them. We use the values of

$$\Delta \equiv (\langle z^2\rangle - \langle z\rangle^2)/\langle z^2\rangle \qquad (3.20)$$

reported by Lücke for this purpose. Following this procedure, we have plotted $\langle z\rangle$ and $\langle z^2\rangle/r$ versus r in Fig. 2 for a set of D values. We find that $\langle E\rangle$ is slightly sensitive to the value of D selected, but $\langle z\rangle$ and $\langle z^2\rangle$ are fairly insensitive to D. The agreement between Fig. 2 and the corresponding figure in Ref. 6 is excellent. This shows that the SDE (2.12) is consistent with the results from the exact dynamics.

An important dynamical measure in the Lorenz dynamics is the average time lapse between two consecutive "flip-flop" transitions. This corresponds in the oscillator analogy to the mean first passage time from the interior of one stable region where the energy envelope is negative to the top of the barrier separating the two regions (where the envelope is zero). For a process starting from $E = E_{\min}$ and evolving according to Eq. (3.11) the average time it takes to reach $E = 0$ is given by[18,19]

$$T_1(0) = \int_{E_{\min}}^{0} \frac{[Q(E)]^2\,\phi'(E)}{D\sigma\psi(E)\,W_s(E)}\,dE \qquad (3.21)$$

where

$$Q(E) \equiv \int_{E_{\min}}^{E} W_s(E')\,dE' \qquad (3.22)$$

The main contribution to the integral comes from the integrand near $E \approx 0$, where $Q(E)$ is very close to unity. Therefore $T_1(0)$ may be approximated by

$$T_1(0) = \frac{1}{D\sigma}\int_{E_{\min}}^{0} \frac{\phi'(E)}{\psi(E)\,W_s(E)}\,dE \qquad (3.23)$$

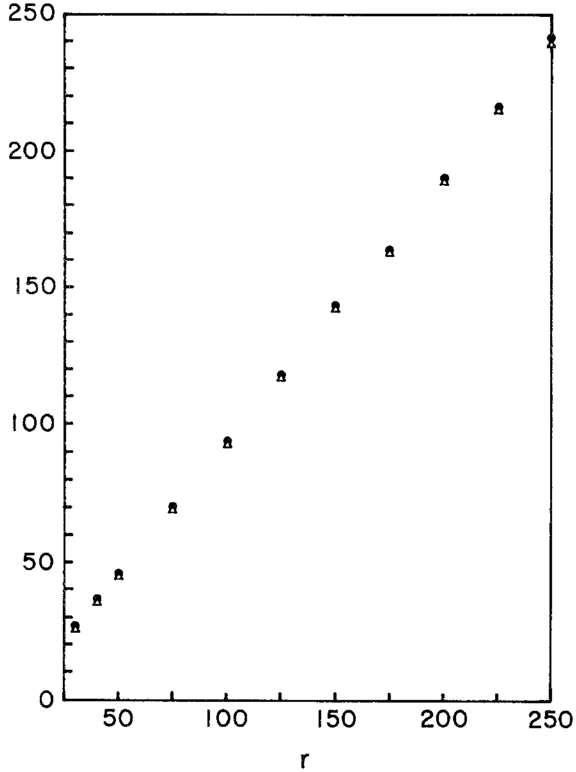

Fig. 2. Theoretical average values of (●) $\langle z \rangle$ and (△) $\langle z^2 \rangle / r$ as a function of r. For each value of r, all values of D between the two curves in Fig. 3 lead to the same value of $\langle z \rangle$ and of $\langle z^2 \rangle / r$ within the visual accuracy of this figure.

Aizawa[8] estimated the mean transition time of the flip-flop process for $r = 28$ and found it to be 1.0/0.44. From Fig. 1 it is seen that four transitions occur within a period of 4.5 units. These results indicate that $T_1(0)$ is of order unity. We calculated $T_1(0)$ for several values of D for each value of r plotted in Fig. 3. This figure shows the range of D values that give rise to $T_1(0)$ between 1 and 10. It is this range of D values that has been used in generating Fig. 2. All D values in this range produce the same plots for Fig. 2.

4. CONCLUSION

In this paper we have explored the viability of the notion of replacing a set of deterministic nonlinear equations having aperiodic solutions with stochastic differential equations. This was first done by Knobloch[9] for the

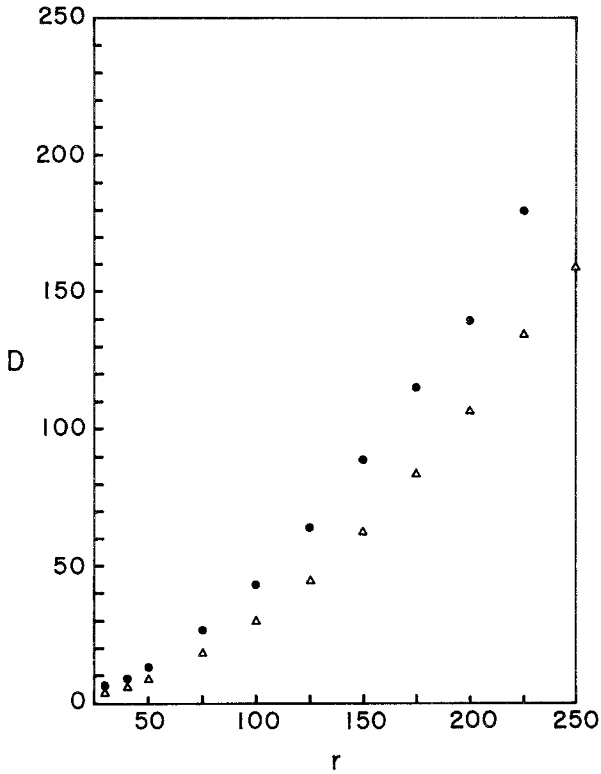

Fig. 3. Values of the mean square level of the fluctuations D and of r that lead to a mean first passage time (●) $T_1(0) = 1.0$ and (△) $T_1(0) = 10.0$.

Lorenz system, in which he replaced the rapidly changing z variable by a stochastic process. The resulting SDE described a linear oscillator with a fluctuating frequency. Knobloch's replacement leads to a process whose stability properties depend sensitively on the choice of parameter values. Pursuing this idea further, we replaced dz/dt by a stochastic process, thereby arriving at a nonlinear SDE for an anharmonic oscillator. This strategy models the Lorenz system by an anharmonic oscillator with a fluctuating frequency in a bistable potential, as given in Eq. (2.12).

The model system shares several qualitative features with the Lorenz equations (2.1). To wit, it retains the steady states and symmetry properties of the latter. The stationary distribution corresponding to the SDE (2.12) reflects the relative time spent by the projected Lorenz trajectory at every point in the x–y plane. We also made a comparison of the energy envelope of the oscillator and the corresponding quantity from the Lorenz model.

The parameter D in Eq. (2.12) [and Eq. (3.2)] can be selected if the persistence times are known for each r value.

Finally, we note that the use of delta-correlated Gaussian fluctuations for $\zeta(t)$ is not justified. Strictly speaking, a simulation of the full Lorenz dynamics should be used to study the statistical properties of $\dot{z}$ (it is insufficient to know the statistics of z) and a simple form for $\zeta(t)$ should be selected based on these properties.

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NOTE ADDED IN PROOF

Related work on the same problem but from a different perspective can be found in C. Nicolis and G. Nicolis, *Phys. Rev. A* **34**:2384 (1986).

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